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STRONG RAMSEY PROPERTIES OF SIMPLICES

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ABSTRACT

In this paper we will show that every simplex X with circumradius ϱ satisfies the following geometric partition property, which proves a conjecture from [FR90].

For every positive real δ there exists a positive real σ such that every χ-colouring of the n-dimensional sphere of radius $\varrho+\delta$ with $\chi \leq (1+\sigma)^n$ results in a monochromatic copy of X.

1. Introduction

In this section we first introduce a few, related geometrical concepts and its history before we state the main result in section 1.4. Furthermore, we will outline the organisation of this paper in section 1.5.

1.1. RAMSEY SETS. In a series of papers Erdős et al. [EGM+73, EGM+75a, EGM+75b] introduced and investigated the following concept.

Definition 1.1: A subset X of the d-dimensional Euclidean space $\mathbf{R}^d$ is called **Ramsey** if for every $\chi \geq 2$ there exists an integer $n = n(X, \chi)$ such that for every χ-colouring of the points of $\mathbf{R}^n$ there exists a monochromatic subset $X' \subseteq \mathbf{R}^n$ congruent to X.

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Erdős et al. [EGM+73] have shown that all Ramsey sets are spherical, that is, every Ramsey set is contained in an appropriate sphere. On the other hand, they also proved that vertex sets of d-dimensional boxes (i.e., the vertex set of rectangular parallelepipeds) are Ramsey. Since then, the list of Ramsey sets was extended; first it was shown in [FR90] that any simplex (i.e., $d+1$ points spanning $\mathbf{R}^d$) is Ramsey. In [Kří91] Kříž proved that if X has a solvable, transitive automorphism group, then X is Ramsey.

The fundamental problem to characterise Ramsey sets remains, however, unanswered. In [Gra94] R. L. Graham conjectured that all spherical sets are Ramsey and offered \$1000 for the solution.

1.2. Sphere Ramsey sets. In [Gra85], R. L. Graham introduced a concept stronger than being Ramsey.

Definition 1.2: A subset X of $\mathbf{R}^d$ is called **sphere Ramsey** if for every $\chi \geq 2$ there exists an integer $n = n(X, \chi)$ and a positive real $\varrho = \varrho(X, \chi)$ such that for any χ-colouring of the points of the sphere $S(\varrho, n) = \{x \in \mathbf{R}^n \colon \|x\| = \varrho\}$ there exists a monochromatic subset $X' \subseteq S(\varrho, n)$ congruent to X.

For a spherical set X let $\varrho(X)$ denote its **circumradius**, i.e., the radius of the smallest sphere containing X.

In [Gra85] R. L. Graham proved that boxes are sphere Ramsey and he asked if one can choose, in Definition 1.2, $\varrho = \varrho(X) + \delta$ for an arbitrary small $\delta > 0$. This was shown to be true in [Fra87]. The following related result for X being a simplex was proved in [MR95].

Theorem 1.3: *Let X be a simplex with circumradius $\varrho(X) = \varrho$. Then for every $\chi \geq 2$ and every real $\delta > 0$ there exists an integer $n = n(X, \chi, \delta)$ such that for any χ-colouring of the points of the sphere $S(\varrho + \delta, n)$ there exists a monochromatic subset $X' \subseteq S(\varrho + \delta, n)$ congruent to X.*

1.3. Exponentially Ramsey sets. Another area of investigation was to study how large the minimum $n = n(X, \chi)$ from Definition 1.1 is. The special case that X consists of two points was proposed by Hadwiger and Nelson. In [Had61] the question for determining the chromatic number $\chi(n)$ of the Euclidean space $\mathbf{R}^n$ was raised, i.e., what is the maximum integer $\chi(n)$ such that for every real $\Delta > 0$ and every $(\chi(n) - 1)$-colouring of the points of $\mathbf{R}^n$ there are two monochromatic points with distance precisely Δ. It was proved in [Had61], [MM61], and [Woo73] that $4 \leq \chi(2) \leq 7$. The current bounds for $\chi(n)$ are

$$(1.2\ldots)^n \leq \chi(n) \leq (3 + o(1))^n.$$

The first exponentially growing lower bound was proved by Frankl and Wilson in [FW81]. The base of the exponential lower bound was recently improved by Raĭgorodskiĭ in [Raĭ00, Raĭ01]. The upper bound was shown by Larman and Rogers in [LR72].

Extending this phenomenon to sets X consisting of more than two points we introduce the following concept.

Definition 1.4: A subset X of $\mathbf{R}^d$ is called **exponentially Ramsey** if there exists a positive real $\sigma = \sigma(X)$ such that for every integer $n \geq d$ and every χ-colouring of the points of $\mathbf{R}^n$ with $\chi \leq (1+\sigma)^n$ there exists a monochromatic subset $X' \subseteq \mathbf{R}^n$ congruent to X.

In other words, X is exponentially Ramsey if the chromatic number of the hypergraph with vertex set $\mathbf{R}^n$ and edges formed by congruent copies of X grows exponentially with n.

It was proved in [FR90] that boxes and simplices are exponentially Ramsey.

1.4. Strong Ramsey sets. The following definition combines the concepts considered in sections 1.2 and 1.3.

Definition 1.5: A subset X of $\mathbf{R}^d$ with circumradius $\varrho(X) = \varrho$ is called **strong Ramsey** if for every real $\delta > 0$ there exists a positive real $\sigma = \sigma(X)$ such that for every integer $n \geq d$ and every χ-colouring of the points of the sphere $S(\varrho+\delta, n)$ with $\chi \leq (1+\sigma)^n$ there exists a monochromatic subset $X' \subseteq S(\varrho+\delta, n)$ congruent to X.

From results in [FW81] and [FR90] it follows that boxes are strong Ramsey (see also section 3.1). Present knowledge, however, does not exclude the possibility that all spherical sets are strong Ramsey. A first step toward this problem is to answer the question of whether obtuse triangles are strong Ramsey. The main purpose of this paper is to answer this question positively and to extend Theorem 1.3 in the sense that it remains true if χ grows exponentially with n (i.e., $\chi \leq (1+\sigma)^n$, where $\sigma = \sigma(X) > 0$). More precisely we will prove the following.

Theorem 1.6: *Every simplex is strong Ramsey.*

1.5. Organisation of the paper. This paper is organised as follows. In section 2 we state some already known results, which were proved in earlier papers. Then in section 3 we introduce the concept of hyper Ramsey sets. This concept is stronger than strong Ramsey and, in fact, later we prove Theorem 3.3, which claims that every simplex is hyper Ramsey. In sections 3.2–3.4 we develop some tools about hyper Ramsey sets. These lemmas simplify the proof of the main result. Finally, the proof of Theorem 3.3 which implies Theorem 1.6 is given in section 4.

2. Preliminary facts

In this section we review a few previously proved results that will be useful in sections 3 and 4.

2.1. EMBEDDING OF FINITE METRIC SPACES. First, we state a well known result that characterises finite metric spaces, which are embeddable into the Euclidean space.

Let $M = (m_{ij})_{i,j=1}^{d+1}$ be a symmetric real matrix with zeros on the diagonal. Then M is said to be of negative type if

$$\sum_{i=1}^{d} \sum_{j=i+1}^{d+1} m_{ij}\zeta_i\zeta_j \leq 0 \tag{1}$$

holds for all choices of $\zeta_1, \zeta_2, \ldots, \zeta_{d+1}$ with $\sum_{i=1}^{d+1} \zeta_i = 0$ and $\sum_{i=1}^{d+1} \zeta_i^2 = 1$.

The following well-known Theorem is due to I. J. Schoenberg (see [Sch38]).

THEOREM 2.1: *A finite metric space $X = \{x_1, x_2, \ldots, x_{d+1}\}$ with distances d_{ij} between x_i and x_j for $1 \leq i, j \leq d+1$ can be embedded into the Euclidean space $\mathbf{R}^d$ if and only if the matrix M with general entry $m_{ij} = d_{ij}^2$ is of negative type.*

Moreover, the embedded image of X is affine independent if and only if inequality (1) is always strict.

2.2. INTERSECTIONS OF PARTITIONS. Another tool we are going to use is taken from [FR87]. It asserts that every sufficiently large family of $(l_0, l_1, \ldots, l_k)$-partitions of an n-element set contains r partitions intersecting in precisely a given pattern.

For positive integers $l_0, l_1, \ldots, l_k$ with $l_0 + l_1 + \cdots + l_k = n$ let $\binom{[n]}{l_0, l_1, \ldots, l_k}$ denote the set of all ordered partitions $\mathcal{A} = (A_0, A_1, \ldots, A_k)$ of $[n] = \{1, 2, \ldots, n\}$ with $|A_i| = l_i$. Obviously, the number of such partitions is

$$\left|\binom{[n]}{l_0, l_1, \ldots, l_k}\right| = \binom{n}{l_0, l_1, \ldots, l_k} = \frac{n!}{l_0! l_1! \cdots l_k!}.$$

For r given $(l_0, l_1, \dots, l_k)$-partitions

$$\mathcal{A}^{(1)} = (A_0^{(1)}, A_1^{(1)}, \dots, A_k^{(1)}), \quad \mathcal{A}^{(2)} = (A_0^{(2)}, A_1^{(2)}, \dots, A_k^{(2)}), \quad \dots,$$
$$\mathcal{A}^{(i)} = (A_0^{(i)}, A_1^{(i)}, \dots, A_k^{(i)}), \quad \dots, \quad \mathcal{A}^{(r)} = (A_0^{(r)}, A_1^{(r)}, \dots, A_k^{(r)})$$

consider the $(k+1) \times \dots \times (k+1)$ (r times) array $M = M(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(r)})$ with general entry

$$m_{t_1 t_2 \dots t_r} = |A_{t_1}^{(1)} \cap A_{t_2}^{(2)} \cap \dots \cap A_{t_r}^{(r)}|, \tag{2}$$

for $0 \leq t_1, t_2, \dots, t_r \leq k$. Observe that for a fixed $0 \leq i \leq k$ and $1 \leq j \leq r$

$$\sum_{t_1, \dots, t_r} \{m_{t_1 t_2 \dots t_r} \colon t_j = i\} = |A_i^{(j)}| = l_i.$$

In [FR87] we proved the following result.

THEOREM 2.2: *Let r and k be positive integers. Then for every real $\lambda > 0$ there exists a real $\varepsilon = \varepsilon(\lambda) > 0$ such that for every positive integer n the following holds:*

If $l_0, l_1, \dots, l_k$ are positive integers with $\sum_{i=0}^{k} l_i = n$ and M is a $(k+1) \times (k+1) \times \dots \times (k+1)$ (r times) array satisfying

(i) *$m_{t_1 t_2 \dots t_r} \geq \lambda n$ for any $0 \leq t_0, t_1, \dots, t_r \leq k$ and*

(ii) *$\sum_{t_1, \dots, t_r} \{m_{t_1 t_2 \dots t_r} \colon t_j = i\} = l_i$ for $i = 0, 1, \dots, k$, and $1 \leq j \leq r$*

then for every $\mathcal{K} \subseteq \binom{[n]}{l_0, l_1, \dots, l_k}$ satisfying

(iii) *$|\mathcal{K}| \geq (1-\varepsilon)^n \binom{n}{l_0, l_1, \dots, l_k}$*

there exists $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(r)} \in \mathcal{K}$ such that

$$M(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(r)}) = M.$$

2.3. APPROXIMATION OF SPHERICAL SETS. In this section we consider a result from [MR95]. This lemma roughly says that for every d and η there exist s, k, a unit vector $a = (a_1, a_2, \dots, a_k)$ and a unit sphere $S(1, d)$ in $\mathbf{R}^s$ such that every z in that sphere can be η approximated by some y in $\mathbf{R}^s$, whose only nonzero entries are $a_1, a_2, \dots, a_k$.

More precisely, for $\mathcal{Z}$ a linear subspace of $\mathbf{R}^s$ let $S(\mathcal{Z}) = S(1, s) \cap \mathcal{Z}$ be the set of all unit vectors in $\mathcal{Z}$. Let $E_s = (e_1, e_2, \dots, e_s)$ denote an orthonormal basis of $\mathbf{R}^s$. Furthermore, let $a = (a_1, a_2, \dots, a_k) \in \mathbf{R}^k$ be a k-dimensional vector and let $K = \{u_1, u_2, \dots, u_k\}$ be a k-element subset of $[s]$ with $u_1 < u_2 < \dots < u_k$.

We will need the following definition,

$$\operatorname{spread}(a, K) = \sum_{j=1}^{k} a_j e_{u_j}.$$

Furthermore, let $\widetilde{K}_1, \widetilde{K}_2, \ldots, \widetilde{K}_d \in [s]^k$ be disjoint sets each of cardinality k such that $\widetilde{K}_1 < \widetilde{K}_2 < \cdots < \widetilde{K}_d$ (here $\widetilde{K} < \widetilde{K}'$ means that all elements of $\widetilde{K}$ are smaller than any element of $\widetilde{K}'$).

We set

$$z_i = z_i(a, \widetilde{K}_i) = \mathrm{spread}(a, \widetilde{K}_i)$$

for $1 \leq i \leq d$ and we denote by

$$\mathcal{Z} = \mathcal{Z}(a, \widetilde{K}_1, \widetilde{K}_2, \ldots, \widetilde{K}_d) = \mathrm{span}(\{z_1, z_2, \ldots, z_d\})$$

the vector space spanned by $z_1, z_2, \ldots, z_d$. Let $[I]^k$ denote the set of all k-element subsets of a set I.

The following lemma was proved in [MR95].

Lemma 2.3: *For every real $\eta > 0$ and every integer d, there exist integers s, k, and a k-dimensional unit vector $a \in S(1,k)$, such that for some $\widetilde{K}_1 < \widetilde{K}_2 < \cdots < \widetilde{K}_d$, $\widetilde{K}_i \in [s]^k$, $i = 1, 2, \ldots, d$ the linear space*

$$\mathcal{Z} = \mathcal{Z}(a, \widetilde{K}_1, \widetilde{K}_2, \ldots, \widetilde{K}_d)$$

has the following property:

For every $z \in S(\mathcal{Z})$ there exists $K \in [s]^k$ such that for $y = \mathrm{spread}(a, K)$,

$$\mathrm{d}(z, y) \leq \eta$$

holds, where $\mathrm{d}(z, y)$ denotes the Euclidean distance between z and y in $\mathbf{R}^s$.

3. Preliminary lemmas

In this section we introduce the concept of hyper Ramsey sets and we will prove a few, somewhat technical lemmas which will simplify the proof of the main result, Theorem 1.6.

3.1. Hyper Ramsey sets. The following concept of hyper Ramsey sets, which was already introduced in [FR90], is stronger but more technical than the concept of strong Ramsey sets.

Definition 3.1: Let $\alpha > 0$ be a real number. A subset X of $\mathbf{R}^d$ with circumradius $\varrho(X) = \varrho$ is called α-**hyper Ramsey** if there exist reals $c = c(X, \alpha)$, $\varepsilon = \varepsilon(X, \alpha) > 0$, and an integer $m_0 = m_0(\alpha)$ such that for every $m \geq m_0$ there exist a finite subset $\mathcal{H} = \mathcal{H}(m) \subseteq \mathbf{R}^m$ satisfying

(i) $\mathcal{H}(m) \subseteq S(\sqrt{\varrho^2 + \alpha}, m)$,

(ii) $|\mathcal{H}(m)| < c^m$, and
(iii) if $\mathcal{K} \subseteq \mathcal{H}(m)$ and $|\mathcal{K}| \geq (1-\varepsilon)^m |\mathcal{H}(m)|$, then there exists a subset $X' \subseteq \mathcal{K}$ congruent to X.

Furthermore, X is called **hyper Ramsey** if X is α-hyper Ramsey for every real $\alpha > 0$.

For sets $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$ consider their **product**

$$X * Y = \{x * y : x \in X, y \in Y\}$$

where $x * y = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$.

It follows from [FW81] that sets of cardinality two are hyper Ramsey (see also [Gra83, Röd83]). Moreover, it was shown in [FR90] that the product of two hyper Ramsey sets is hyper Ramsey. Both results together imply the following theorem.

THEOREM 3.2: *Every box is hyper Ramsey.*

Again, stressing the fact that if X is hyper Ramsey then X is also strong Ramsey, we observe that every box is strong Ramsey as mentioned in section 1.4. By the same reason the main result of this paper, Theorem 1.6, is a consequence of the following theorem.

THEOREM 3.3: *Every simplex is hyper Ramsey.*

The rest of this paper is devoted to the proof of Theorem 3.3.

3.2. PRODUCTS OF α-HYPER RAMSEY SETS. The following "product result", which is needed in our proof of the main result, is a modification of Theorem 2.2 in [FR90].

LEMMA 3.4: *Let $V \subseteq \mathbf{R}^{d_1}$ be a finite, α^V-hyper Ramsey set with $\varrho(V) = \varrho^V$ and let $T \subseteq \mathbf{R}^{d_2}$ be a finite, α^T-hyper Ramsey set with $\varrho(T) = \varrho^T$. Then $V * T$ is α^{V*T}-hyper Ramsey for $\alpha^{V*T} = \alpha^V + \alpha^T$.*

Proof: Let c^V, ε^V, m_0^V be constants, and let $\mathcal{H}^V(m)$ for each $m \geq m_0^V$ be sets witnessing that V is α^V-hyper Ramsey (see Definition 3.1). In the same way, let c^T, ε^T, m_0^T, and $\mathcal{H}^T(m)$ for each $m \geq m_0^T$ correspond to T. Set τ to the solution of $(c^T)^{(|T|-1)\tau} = (1-\varepsilon^V)^{-1/2}$, let $m_0^{V*T} = m_0^{V*T}(V, \alpha^V, T, \alpha^T)$ be sufficiently large and $\widetilde{n} = n + \lfloor \tau n \rfloor \geq m_0^{V*T}$. We will show that $\mathcal{H}^{V*T}(\widetilde{n}) = \mathcal{H}^V(n) * \mathcal{H}^T(\lfloor \tau n \rfloor)$ witnesses that $V * T$ is α^{V*T}-hyper Ramsey.

Clearly, $\varrho^{V*T} = \varrho(V * T) = \sqrt{(\varrho^V)^2 + (\varrho^T)^2}$ and

$$\mathcal{H}^{V*T}(\tilde{n}) \subseteq S\Big(\sqrt{(\varrho^V)^2 + \alpha^V + (\varrho^T)^2 + \alpha^T}, \tilde{n}\Big) = S\Big(\sqrt{(\varrho^{V*T})^2 + \alpha^{V*T}}, \tilde{n}\Big)$$

shows that (i) of Definition 3.1 is satisfied. Suppose that $\mathcal{X} = \mathcal{X}(\tilde{n}) \subseteq \mathcal{H}^{V*T}(\tilde{n})$ does not contain a copy of $V * T$. For each $v \in \mathcal{H}^V(n)$ consider the set $\mathcal{X}_v = \mathcal{X} \cap (\{v\} * \mathcal{H}^T(\lfloor \tau n \rfloor))$. $\mathcal{X}_v$ is congruent to a subset of $\mathcal{H}^T(\lfloor \tau n \rfloor)$ and therefore $\mathcal{X}_v$ contains more than $|\mathcal{X}_v| - (1 - \varepsilon^T)^{\lfloor \tau n \rfloor} |\mathcal{H}^T(\lfloor \tau n \rfloor)|$ copies of T. Now we estimate the number $\mathcal{X}(T)$ of copies T^* of T such that $T^* \subseteq \mathcal{X}$ and $T^* = v * \tilde{T}$ for some $v \in \mathcal{H}^V(n)$ and $\tilde{T}$ a copy of T in $\mathcal{H}^T(\lfloor \tau n \rfloor)$:

$$\begin{aligned} \mathcal{X}(T) \geq & \sum_{v \in \mathcal{H}^V(n)} (|\mathcal{X}_v| - (1 - \varepsilon^T)^{\lfloor \tau n \rfloor} \cdot |\mathcal{H}^T(\lfloor \tau n \rfloor)|) \\ = & |\mathcal{X}| - |\mathcal{H}^V(n)||\mathcal{H}^T(\lfloor \tau n \rfloor)|(1 - \varepsilon^T)^{\lfloor \tau n \rfloor}. \end{aligned} \tag{3}$$

On the other hand, for every copy $\tilde{T}$ of T in $\mathcal{H}^T(\lfloor \tau n \rfloor)$ let $\mathcal{V}_{\tilde{T}} = \{v \in \mathcal{H}^V(n) \colon \{v\} * \tilde{T} \subseteq \mathcal{X}_v\}$. Since $\mathcal{X}$ contains no copy of $V * T$,

$$|\mathcal{V}_{\tilde{T}}| < (1 - \varepsilon^V)^n |\mathcal{H}^V(n)|.$$

This means that the number $\mathcal{X}(T)$ can be bounded from above by

$$\begin{aligned} \mathcal{X}(T) = & \sum \{|\mathcal{V}_{\tilde{T}}| \colon \tilde{T} \text{ is a copy of } T \text{ in } \mathcal{H}^T(\lfloor \tau n \rfloor)\} \\ < & \binom{|\mathcal{H}^T(\lfloor \tau n \rfloor)|}{|T|} |\mathcal{H}^V(n)| \cdot (1 - \varepsilon^V)^n \\ < & (c^T)^{\lfloor \tau n \rfloor (|T| - 1)} |\mathcal{H}^T(\lfloor \tau n \rfloor)||\mathcal{H}^V(n)| \cdot (1 - \varepsilon^V)^n \\ \leq & |\mathcal{H}^T(\lfloor \tau n \rfloor)||\mathcal{H}^V(n)| \cdot (1 - \varepsilon^V)^{n/2}, \end{aligned} \tag{4}$$

where the last inequality follows due to the choice of τ and n.

Combining (3) and (4), we infer that

$$|\mathcal{X}| < |\mathcal{H}^V(n)||\mathcal{H}^T(\lfloor \tau n \rfloor)|((1 - \varepsilon^V)^{n/2} + (1 - \varepsilon^T)^{\lfloor \tau n \rfloor}),$$

and thus

$$|\mathcal{X}| < |\mathcal{H}^V(n)||\mathcal{H}^T(\lfloor \tau n \rfloor)|(1 - \varepsilon^{V*T})^{\tilde{n}} = (1 - \varepsilon^{V*T})^{\tilde{n}} |\mathcal{H}^{V*T}(\tilde{n})|$$

for some appropriately chosen $\varepsilon^{V*T} > 0$ and n sufficiently large, which implies (iii) of Definition 3.1. In order to verify (ii) of Definition 3.1, set

$$c^{V*T} = \max\{c^V, c^T\}.$$

As

$$|\mathcal{H}^{V*T}(\tilde{n})| = |\mathcal{H}^{V}(n)||\mathcal{H}^{T}(\lfloor \tau n \rfloor)| \leq (c^{V})^{n}(c^{T})^{\lfloor \tau n \rfloor} \leq (c^{V*T})^{\tilde{n}},$$

the sequence $\mathcal{H}^{V*T}(\tilde{n}) = \mathcal{H}^{V}(n) * \mathcal{H}^{T}(\lfloor \tau n \rfloor)$ for all $\tilde{n} \geq m_0^{V*T}$ shows that $V * T$ is α^{V*T}-hyper Ramsey. ∎

3.3. α-HYPER RAMSEY SIMPLICES ARE DENSE. The aim of this section is to show that for every simplex $Z = \{z_1, z_2, \dots, z_{d+1}\}$ and for every positive real ϑ there exists an α-hyper Ramsey simplex $V = \{v_1, v_2, \dots, v_{d+1}\}$ such that, for all $1 \leq i, i' \leq d+1$,

$$|\,\mathrm{d}^2(v_i, v_{i'}) - \mathrm{d}^2(z_i, z_{i'})| \leq \vartheta. \tag{5}$$

This is proved in Lemma 3.9.

The construction of V is done in two steps. First, using Lemma 2.3 we find integers s and k, a vector $a = (a_1, a_2, \dots, a_k)$ and $y_{j_i} = \mathrm{spread}(a, K^{(j_i)})$ with $1 \leq j_1, \dots, j_{d+1} \leq \binom{s}{k}$ (where $\{K^{(i)}\colon 1 \leq i \leq \binom{s}{k}\}$ is an enumeration of all k-element subsets of $[s]$) such that

$$|\,\mathrm{d}^2(y_{j_i}, y_{j_{i'}}) - \mathrm{d}^2(z_i, z_{i'})| \quad \text{is “small” for every } 1 \leq i, i' \leq d+1. \tag{6}$$

The aim of the second step is to construct an α-hyper Ramsey simplex $V = \{v_1, v_2, \dots, v_{d+1}\}$ such that

$$|\,\mathrm{d}^2(v_i, v_{i'}) - \mathrm{d}^2(y_{j_i}, y_{j_{i'}})| \quad \text{is “small” for every } 1 \leq i, i' \leq d+1. \tag{7}$$

For this, we associate v_i with a conveniently chosen partition of $[n]$ for some sufficiently large n. For a partition $\mathcal{A} = (A_0, A_1, \dots, A_k)$ of $[n]$ and the vector $a = (a_1, a_2, \dots, a_k)$ we will consider the n-dimensional vector $v^{\mathcal{A}} = (\xi_1^{\mathcal{A}}, \xi_2^{\mathcal{A}}, \dots, \xi_n^{\mathcal{A}})$ defined as follows,

$$\xi_t^{\mathcal{A}} = \begin{cases} 0 & \text{if } t \in A_0, \\ a_j/\sqrt{l} & \text{if } t \in A_j. \end{cases}$$

The aim of the next lemma, Lemma 3.5, is to construct a family

$$\mathfrak{A} = \{\mathcal{A}^{(i)} = (A_0^{(i)}, A_1^{(i)}, \dots, A_k^{(i)})\colon 1 \leq i \leq r\}$$

of partitions of $[n]$ such that

$$|\,\mathrm{d}^2(v^{\mathcal{A}^{(i)}}, v^{\mathcal{A}^{(i')}}) - \mathrm{d}^2(y_i, y_{i'})| \leq 4\frac{n - ls}{l(k+1)} \tag{8}$$

for all $1 \leq i, i' \leq \binom{s}{k}$. Setting $v_i = v^{\mathcal{A}^{(j_i)}}$ and choosing n and l appropriately will imply (7). Then (6) combined with (7) yields (5).

Next we will formulate Lemma 3.5 in which we will work with the following set-up:

(I) l, s, k, and n are integers, set

$$r = \binom{s}{k}, \quad \lambda = \frac{n - ls}{n(k+1)^r}, \quad \text{and}$$

$$l_j = \lambda n(k+1)^{r-1} + \begin{cases} (s-k)l & \text{if } j = 0, \\ l & \text{if } j = 1, 2, \ldots, k, \end{cases}$$

(II) $\{K^{(i)}\colon 1 \le i \le r\}$ is the family of all k-element subsets of $[s]$,

(III) $\mathfrak{A} = \{\mathcal{A}^{(i)} = \mathcal{A}^{(i)}(K^{(i)}) = (A_0^{(i)}, A_1^{(i)}, \ldots, A_k^{(i)})\colon 1 \le i \le r\}$ is a family of $(l_0, l_1, \ldots, l_k)$-partitions of $[n]$, where $\mathcal{A}^{(i)}$ will depend on $K^{(i)}$,

(IV) $a = (a_1, a_2, \ldots, a_k) \in S(1, k)$ is a unit vector and we define for $i = 1, 2, \ldots, r$ the n-dimensional vector $v^{\mathcal{A}^{(i)}} = (\xi_1^{\mathcal{A}^{(i)}}, \xi_2^{\mathcal{A}^{(i)}}, \ldots, \xi_n^{\mathcal{A}^{(i)}})$ by

$$\xi_t^{\mathcal{A}^{(i)}} = \begin{cases} 0 & \text{if } t \in A_0, \\ a_j/\sqrt{l} & \text{if } t \in A_j, \end{cases} \quad \text{and}$$

(V) set $y_i = \operatorname{spread}(a, K^{(i)})$ for $i = 1, 2, \ldots, r$.

The next lemma ensures the existence of a family $\mathfrak{A}$ satisfying (8).

Lemma 3.5: *Let l, s, k, and n be integers such that $n > ls$ and $(k+1)^{\binom{s}{k}}$ divides $n - ls$. Then there exists a family of $(l_0, l_1, \ldots, l_k)$-partitions of $[n]$,*

$$\mathfrak{A} = \{\mathcal{A}^{(i)} = \mathcal{A}^{(i)}(K^{(i)})\colon 1 \le i \le r\},$$

such that for every $a = (a_1, a_2, \ldots, a_k) \in S(1, k)$ the following holds:

$$m_{t_1 t_2 \cdots t_r} = |A_{t_1}^{(1)} \cap A_{t_2}^{(2)} \cap \cdots \cap A_{t_r}^{(r)}| \ge \lambda n \quad \text{for } 0 \le t_1, t_2, \ldots, t_r \le k, \tag{9}$$

$$\{v^{\mathcal{A}}\colon \mathcal{A} \in \mathfrak{A}\} \text{ is an affine independent set, and} \tag{10}$$

for every $1 \le i \ne i' \le r$,

$$|\,\mathrm{d}^2(v^{\mathcal{A}^{(i)}}, v^{\mathcal{A}^{(i')}}) - \mathrm{d}^2(y_i, y_{i'})| \le 4\frac{n - ls}{l(k+1)}. \tag{11}$$

Proof: Given integers l, s, k, and n satisfying the assumptions of the lemma, first, we will construct a family $\mathfrak{A}$ consisting of $(l_0, l_1, \ldots, l_k)$-partitions $\mathcal{A}^{(i)} = (A_0^{(i)}, A_1^{(i)}, \ldots, A_k^{(i)})$ for $1 \le i \le r$ satisfying (9).

Let $\{K^{(i)} = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)}\}\colon 1 \le i \le r\}$ be an enumeration of all k-element subsets of $[s]$ with $u_1^{(i)} < u_2^{(i)} < \cdots < u_k^{(i)}$. Furthermore, let $L_1, L_2, \ldots, L_s$ be pairwise disjoint sets, each of size l. For each $i = 1, 2, \ldots, r$ we define a different partition $(B_0^{(i)}, B_1^{(i)}, \ldots, B_k^{(i)})$ of $\bigcup_{t=1}^{s} L_t$ by

$$B_0^{(i)} = \bigcup_{t \in [s] \setminus K^{(i)}}^{s} L_t,$$

$$B_1^{(i)} = L_{u_1^{(i)}}, \qquad B_2^{(i)} = L_{u_2^{(i)}}, \quad \ldots, \quad B_k^{(i)} = L_{u_k^{(i)}}.$$

For each r-tuple $(j_1, j_2, \ldots, j_r)$ with $0 \leq j_i \leq k$ let $C_{j_1 j_2 \cdots j_r}$ be a set of cardinality λn (which is an integer by the assumptions of the lemma) and let $C_{j_1 j_2 \cdots j_r}$ and $C_{j'_1 j'_2 \cdots j'_r}$ be disjoint whenever the r-tuples $(j_1, j_2, \ldots, j_r)$ and $(j'_1, j'_2, \ldots, j'_r)$ differ in at least one entry. We now define $\mathfrak{A}$ by setting

$$A_j^{(i)}(K^{(i)}) = A_j^{(i)} = B_j^{(i)} \cup \bigcup_{j_i = j} C_{j_1 j_2 \cdots j_r}, \tag{12}$$

where the union is taken over all $(k+1)^{r-1}$ different r-tuples $(j_1, j_2, \ldots, j_r)$ with j_i being fixed.

Claim 3.6: *Let $\mathfrak{A}$ be defined as in (12). Then*

(i) $\mathfrak{A} \subseteq \binom{[n]}{l_0, l_1, \ldots, l_k}$ *and*
(ii) *inequality (9) holds.*

Proof of Claim 3.6: Note that $\mathcal{A}^{(i)} = (A_0^{(i)}, A_1^{(i)}, \ldots, A_k^{(i)})$ forms a partition of

$$N = \bigcup_{t=1}^{s} L_t \cup \bigcup_{(j_1, j_2, \ldots, j_r)} C_{j_1 j_2 \cdots j_r}$$

for each $i = 1, 2, \ldots, r$. Clearly

$$\begin{aligned}
|N| &= ls + \lambda n (k+1)^r = n, \\
|A_0^{(i)}| &= |B_0^{(i)}| + \Big| \bigcup_{j_i = 0} C_{j_1 j_2 \cdots j_r} \Big| \\
&= (s-k)l + \lambda n (k+1)^{(r-1)} = l_0, \quad \text{and} \\
|A_j^{(i)}| &= |B_j^{(i)}| + \Big| \bigcup_{j_i = j} C_{j_1 j_2 \cdots j_r} \Big| \\
&= l + \lambda n (k+1)^{(r-1)} = l_j \quad \text{for each } j = 1, 2, \ldots, k
\end{aligned}$$

which implies (i) of Claim 3.6.

Also note that given $0 \leq t_1, t_2, \ldots, t_r \leq k$,

$$A_{t_i}^{(i)} \supseteq \bigcup_{j_i = t_i} C_{j_1 j_2 \cdots j_r} \supseteq C_{t_1 t_2 \cdots t_r} \quad \text{for each } i = 1, 2, \ldots, r,$$

and consequently

$$m_{t_1 t_2 \cdots t_r} = |A_{t_1}^{(1)} \cap A_{t_2}^{(2)} \cap \cdots \cap A_{t_r}^{(r)}| \geq |C_{t_1 t_2 \cdots t_r}| = \lambda n$$

holds, which yields (9). ∎

It is easy to see that (10) holds. In fact, since a is a unit vector, $a_q \neq 0$ for some $1 \leq q \leq k$. Let $\nu_1, \nu_2, \ldots, \nu_r$ be reals such that $\sum_i \nu_i v^{\mathcal{A}^{(i)}} = 0$. For each $i = 1, 2, \ldots, r$ we are going to fix $1 \leq x_i \leq n$ such that the x_i-th coordinate of $v^{\mathcal{A}^{(i')}}$ is not equal to 0 if and only if $i = i'$.

For $i = 1, 2, \ldots, r$ we consider the set $C_i = C_{j_1 j_2 \cdots j_r}$, where $j_i = q$ and all the other indices are 0. Observe that for such a set

$$C_i = C_{0,0,\ldots,0,q,0,\ldots,0} \subseteq \begin{cases} A_q^{(i')} & \text{if } i' = i, \\ A_0^{(i')} & \text{if } i' \neq i. \end{cases}$$

In particular, for every $x \in C_i$ the x-th coordinate of the vector $v^{\mathcal{A}^{(i')}}$ satisfies

$$(v^{\mathcal{A}^{(i')}})_x = \begin{cases} \frac{a_q}{\sqrt{l}} \neq 0 & \text{if } i' = i \\ 0 & \text{if } i' \neq i \end{cases}$$

and therefore for every $i = 1, 2, \ldots, r$ there is a $1 \leq x_i \leq n$ such that $(v^{\mathcal{A}^{(i')}})_{x_i} \neq 0$ for $v^{\mathcal{A}^{(i)}}$ only. This implies $\nu_i = 0$ for each $i = 1, 2, \ldots, r$. In other words, $\{v^{\mathcal{A}} \colon \mathcal{A} \in \mathfrak{A}\}$ is a linearly and therefore affine independent set.

In order to prove inequality (11), we need to calculate the cardinalities of intersections of $A_j^{(i)} \cap A_{j'}^{(i')}$. These cardinalities will depend on $u_j^{(i)}$ and $u_{j'}^{(i')}$. We summarise these straightforward calculations in the following claim.

Claim 3.7: *Let $A_j^{(i)}$ for $j = 0, 1, \ldots, k$ and $i = 1, 2, \ldots, r$ be defined as in (12). Then for $1 \leq j, j' \leq k$ and $1 \leq i \neq i' \leq r$,*

$$|A_j^{(i)} \cap A_{j'}^{(i')}| = \begin{cases} \lambda n (k+1)^{r-2} & \text{if } u_j^{(i)} \neq u_{j'}^{(i')} \\ \lambda n (k+1)^{r-2} + l & \text{if } u_j^{(i)} = u_{j'}^{(i')} \end{cases} \tag{13}$$

and

$$|A_j^{(i)} \cap A_0^{(i')}| = \begin{cases} \lambda n (k+1)^{r-2} + l & \text{if } u_j^{(i)} \notin K^{(i')}, \\ \lambda n (k+1)^{r-2} & \text{if } u_j^{(i)} \in K^{(i')}. \end{cases} \tag{14}$$

Proof of Claim 3.7: First, suppose $1 \leq j, j' \leq k$, and $K^{(i)} = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)}\}$, $K^{(i')} = \{u_1^{(i')}, u_2^{(i')}, \ldots, u_k^{(i')}\}$ are given and $u_j^{(i)} \neq u_{j'}^{(i')}$ holds. This implies $B_j^{(i)} \cap B_{j'}^{(i')} = \emptyset$, and therefore

$$|A_j^{(i)} \cap A_{j'}^{(i')}| = \left| \bigcup_{j_i = j, j_{i'} = j'} C_{j_1 j_2 \cdots j_r} \right| = \lambda n (k+1)^{r-2}.$$

Now, if on the other hand $u_j^{(i)} = u_{j'}^{(i')}$, then $B_j^{(i)} = B_{j'}^{(i')} = L_{u_j^{(i)}}$ which yields

$$|A_j^{(i)} \cap A_{j'}^{(i')}| = |L_{u_j^{(i)}}| + \left| \bigcup_{j_i = j, j_{i'} = j'} C_{j_1 j_2 \cdots j_r} \right| = l + \lambda n(k+1)^{r-2}.$$

Next we want to calculate $|A_j^{(i)} \cap A_0^{(i')}|$ for $j \geq 1$ and $i \neq i'$. If $u_j^{(i)} \notin K^{(i')}$ then $L_{u_j^{(i)}} \subseteq A_j^{(i)} \cap A_0^{(i')}$ and thus

$$|A_j^{(i)} \cap A_0^{(i')}| = |L_{u_j^{(i)}}| + \left| \bigcup_{j_i = j, j_{i'} = j'} C_{j_1 j_2 \cdots j_r} \right| = l + \lambda n(k+1)^{r-2}.$$

Finally, $u_j^{(i)} \in K^{(i')}$ implies

$$|A_j^{(i)} \cap A_0^{(i')}| = \left| \bigcup_{j_i = j, j_{i'} = j'} C_{j_1 j_2 \cdots j_r} \right| = \lambda n(k+1)^{r-2}. \qquad \blacksquare$$

We now finish the proof of Lemma 3.5 by showing (11). Let $a = (a_1, a_2, \ldots, a_k)$ in $S(1,k)$ be given; for the sake of convenience we set $a_0 = 0$. Consider $K^{(i)}$, $K^{(i')}$ and the corresponding partitions $\mathcal{A}^{(i)} = \mathcal{A}^{(i)}(K^{(i)})$ and $\mathcal{A}^{(i')} = \mathcal{A}^{(i')}(K^{(i')})$. Furthermore, let $v^{\mathcal{A}^{(i)}}$ and $v^{\mathcal{A}^{(i')}}$ be defined as stated above. Having in mind that i and i' (and thus $K^{(i)} = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)}\}$ and $K^{(i')} = \{u_1^{(i')}, u_2^{(i')}, \ldots, u_k^{(i')}\}$) have been fixed we now infer

$$\begin{aligned} \mathrm{d}^2(v^{\mathcal{A}^{(i)}}, v^{\mathcal{A}^{(i')}}) &= \sum_{t=1}^{n} (\xi_t^{\mathcal{A}^{(i)}} - \xi_t^{\mathcal{A}^{(i')}})^2 \\ &= \sum_{j=1}^{k} \sum_{j'=1}^{k} \left|A_j^{(i)} \cap A_{j'}^{(i')}\right| \frac{(a_j - a_{j'})^2}{l} \\ &\quad + \sum_{j=1}^{k} \left|A_j^{(i)} \cap A_0^{(i')}\right| \frac{a_j^2}{l} + \sum_{j'=1}^{k} \left|A_0^{(i)} \cap A_{j'}^{(i')}\right| \frac{a_{j'}^2}{l} \\ &= \sum_{j,j' \geq 1} \{(a_j - a_{j'})^2 \colon u_j^{(i)} = u_{j'}^{(i')}\} \\ &\quad + \sum_{j \geq 1} \{a_j^2 \colon u_j^{(i)} \notin K^{(i')}\} + \sum_{j' \geq 1} \{a_{j'}^2 \colon u_{j'}^{(i')} \notin K^{(i)}\} \\ &\quad + \frac{\lambda n(k+1)^{r-2}}{l} \sum_{j=0}^{k} \sum_{j'=0}^{k} (a_j - a_{j'})^2, \end{aligned} \tag{15}$$

where we used (13) and (14) for the last equality. Finally, let $y_i = \text{spread}(a, K^{(i)})$ and $y_{i'} = \text{spread}(a, K^{(i')})$; then

$$\begin{aligned} \mathrm{d}^2(y_i, y_{i'}) = &\sum_{j,j' \geq 1} \{(a_j - a_{j'})^2 \colon u_j^{(i)} = u_{j'}^{(i')}\} \\ &+ \sum_{j \geq 1} \{a_j^2 \colon u_j^{(i)} \notin K^{(i')}\} + \sum_{j' \geq 1} \{a_{j'}^2 \colon u_{j'}^{(i')} \notin K^{(i)}\}. \end{aligned} \tag{16}$$

Before we finally prove (11), we derive the following (easily provable, but not the best possible) bound,

$$\sum_{j=0}^{k} \sum_{j'=0}^{k} (a_j - a_{j'})^2 = \sum_{b=0}^{k} \sum_{j'=0}^{k} (a_{(j'+b) \bmod k+1} - a_{j'})^2 \leq 4(k+1),$$

from the fact that $a = (a_0, a_1, \ldots, a_k)$ has length 1. We finish the proof and infer (11) from (15), (16), the bound above, and $\lambda n = (n - ls)/(k+1)^r$,

$$|\,\mathrm{d}^2(v^{\mathcal{A}^{(i)}}, v^{\mathcal{A}^{(i')}}) - \mathrm{d}^2(y_i, y_{i'})| = \frac{\lambda n (k+1)^{r-2}}{l} \sum_{j=0}^{k} \sum_{j'=0}^{k} (a_j - a_{j'})^2 \leq 4 \frac{n - ls}{l(k+1)}. \qquad \blacksquare$$

Remark 3.8: Keeping k, s (and thus r), and $a = (a_1, a_2, \ldots, a_k)$ fixed, we later (see Lemma 3.9) let l and n tend to infinity. The ratio $\lambda n / l$, however, will be a constant independent of l and n (see equality (22)). Consequently, it follows from the right-hand side of (15) that the distances $\mathrm{d}(v^{\mathcal{A}^{(i)}}, v^{\mathcal{A}^{(i')}})$ will be fixed for $1 \leq i, i' \leq k$ as l and n tend to infinity.

We are now able to prove the main lemma of this section.

LEMMA 3.9: *Let $Z = \{z_1, z_2, \ldots, z_{d+1}\}$ be an arbitrary simplex with circumradius $\varrho(Z) = \varrho^Z$ and let $\vartheta > 0$ be an arbitrary real. Then there exists a simplex $V = \{v_1, v_2, \ldots, v_{d+1}\}$ with $\varrho(V) \leq \varrho^Z \sqrt{1 + \vartheta/8}$ which is α-hyper Ramsey for*

$$\alpha = (\varrho^Z)^2 (1 + \vartheta/8) - \varrho(V)^2$$

and, moreover, such that

$$|\,\mathrm{d}^2(v_i, v_{i'}) - \mathrm{d}^2(z_i, z_{i'})| \leq \vartheta \tag{17}$$

for all $1 \leq i, i' \leq d+1$.

Proof of Lemma 3.9: Without loss of generality, assume that $\varrho^Z = 1$ and ϑ is rational, $\vartheta = p/q$ with $p, q > 0$. Set $\eta = \vartheta/16$ and apply Lemma 2.3 for η

and d to find s, k, a k-dimensional unit vector $a = (a_1, a_2, \ldots, a_k) \in S(1,k)$, and k-element sets $\widetilde{K}_1 < \widetilde{K}_2 < \cdots < \widetilde{K}_d$, with $\widetilde{K}_i \in [s]^k$ for $i = 1, 2, \ldots, d$ (recall $\widetilde{K} < \widetilde{K}'$ means that all elements of $\widetilde{K}$ are smaller than any element of $\widetilde{K}'$).

Without loss of generality (using the notation of section 2.3), assume that $Z \subseteq S(\mathcal{Z})$ where $\mathcal{Z} = \mathcal{Z}(a, \widetilde{K}_1, \widetilde{K}_2, \ldots, \widetilde{K}_d)$. Moreover, by Lemma 2.3, we also find sets $K^{(i_1)}, K^{(i_2)}, \ldots, K^{(i_{d+1})}$ such that for $t = 1, 2, \ldots, d+1$ the set $Y = \{y_1, y_2, \ldots, y_{d+1}\}$ defined by $y_t = \mathrm{spread}(a, K^{(i_t)}) \in S(1,s)$ satisfies

$$\mathrm{d}(z_t, y_t) \leq \eta. \tag{18}$$

Clearly, the following inequality holds for every $1 \leq i, i' \leq d+1$ by (18) and our choice of η:

$$\begin{aligned} |\,\mathrm{d}^2(y_i, y_{i'}) - \mathrm{d}^2(z_i, z_{i'})| &= |\,\mathrm{d}(z_i, z_{i'}) + \mathrm{d}(y_i, y_{i'})| \cdot |\,\mathrm{d}(z_i, z_{i'}) - \mathrm{d}(y_i, y_{i'})| \\ &\leq |\,\mathrm{d}(z_i, z_{i'}) + \mathrm{d}(y_i, y_{i'})| \cdot |\,\mathrm{d}(z_i, y_i) + \mathrm{d}(z_{i'}, y_{i'})| \\ &\leq 4 \cdot 2\eta = 8\eta \\ &= \vartheta/2. \end{aligned} \tag{19}$$

On the other hand, let l be an arbitrary multiple of $\omega = 8q(k+1)^{\binom{s}{k}-1}$ and set

$$n = l\Big(s + \frac{\vartheta(k+1)}{8}\Big). \tag{20}$$

Consequently, $(k+1)^{\binom{s}{k}}$ divides

$$n - ls = \frac{l\vartheta(k+1)}{8}. \tag{21}$$

Hence, l, s, k, and n satisfy the assumptions of Lemma 3.5 and we find a family of partitions

$$\mathfrak{A} = \{\mathcal{A}^{(i)} = \mathcal{A}^{(i)}(K^{(i)}) = (A_0^{(i)}, A_1^{(i)}, \ldots, A_k^{(i)})\colon 1 \leq i \leq \binom{s}{k}\}.$$

From now on we will refer to the set-up (I)–(V) stated before Lemma 3.5. Now, consider the subfamily of partitions, $\{\mathcal{A}^{(i_1)}, \mathcal{A}^{(i_2)}, \ldots, \mathcal{A}^{(i_{d+1})}\} \subseteq \mathfrak{A}$ (associated with $K^{(i_1)}, K^{(i_2)}, \ldots, K^{(i_{d+1})}$), and corresponding vectors (see Lemma 3.5) $v_1 = v^{\mathcal{A}^{(i_1)}}$, $v_2 = v^{\mathcal{A}^{(i_2)}}, \ldots, v_{d+1} = v^{\mathcal{A}^{(i_{d+1})}}$. Lemma 3.5 yields that $V = \{v_1, v_2, \ldots, v_{d+1}\}$ is a simplex. Furthermore, in the notation of Lemma 3.5,

$$\frac{\lambda n}{l} = \frac{n - ls}{l(k+1)^r} = \frac{\vartheta}{8(k+1)^{r-1}} \tag{22}$$

holds and $\lambda n/l$ is independent of l and n. This implies, by Remark 3.8, that the simplex V is independent of l and n. Moreover, (11) and (21) yield

$$|\,\mathrm{d}^2(v_i, v_{i'}) - \mathrm{d}^2(y_i, y_{i'})| \leq 4\frac{n-ls}{l(k+1)} = \frac{\vartheta}{2} \tag{23}$$

for every $1 \leq i, i' \leq d+1$. Notice that the upper bound in (23) is independent of n. Combining (19) and (23) we obtain (17).

Now we are going to show that V is α-hyper Ramsey. This means that for every sufficiently large n we need to show the existence of a set $\mathcal{H}(n)$ satisfying (i)–(iii) of Definition 3.1. We first show the existence of $\mathcal{H}(n)$ for every n satisfying (20) with l an arbitrary multiple of ω.

Consider the family

$$\mathcal{H}(n) = \left\{v^{\mathcal{A}} \colon \mathcal{A} \in \binom{[n]}{l_0, l_1, \ldots, l_k}\right\}$$

of n-dimensional vectors. Again using the notation of Lemma 3.5, by (21) we infer

$$\|v^{\mathcal{A}}\|^2 = \sum_{j=1}^{k} l_j \frac{a_j^2}{l} = \frac{l_1}{l} = 1 + \frac{n-ls}{l(k+1)} = 1 + \frac{\vartheta}{8} \tag{24}$$

for every $v^{\mathcal{A}} \in \mathcal{H}(n)$ and therefore $\mathcal{H}(n) \subseteq S(\sqrt{1+\vartheta/8}, n)$. This verifies (i) of Definition 3.1 for $\varrho^Z = 1$ which we assumed above. If $\varrho^Z \neq 1$ the same calculation yields

$$\mathcal{H}(n) \subseteq S(\varrho^Z\sqrt{1+\vartheta/8}, n). \tag{25}$$

Since $\{v^{\mathcal{A}} \colon \mathcal{A} \in \binom{[n]}{l_0, l_1, \ldots, l_k}\}$ contains $V = \{v_1, v_2, \ldots, v_{d+1}\}$ we have $\varrho(V) \leq \varrho^Z\sqrt{1+\vartheta/8}$. Clearly (25) is equivalent to

$$\mathcal{H}(n) \subseteq S(\sqrt{\varrho(V)^2+\alpha}, n) \quad \text{with } \alpha = (\varrho^Z)^2(1+\vartheta/8) - \varrho(V)^2.$$

Therefore, the property (i) of Definition 3.1 is verfied for every ϱ^Z.

On the other hand,

$$|\mathcal{H}(n)| < (k+1)^n$$

and thus (ii) holds as well. Finally, we will verify property (iii) of Definition 3.1. For λ mentioned above consider $\varepsilon = \varepsilon(\lambda)$ guaranteed by Theorem 2.2 and let $\mathcal{K} \subseteq \mathcal{H}(n)$ be such that $|\mathcal{K}| \geq (1-\varepsilon)^n|\mathcal{H}(n)|$ (i.e., $\mathcal{K}$ satisfies condition (iii) of Theorem 2.2, where we use the natural correspondence between $v^{\mathcal{A}}$ and $\mathcal{A}$

for $v^A \in \mathcal{K}$). Let $M = M(\mathcal{A}^{(i_1)}, \mathcal{A}^{(i_2)}, \dots, \mathcal{A}^{(i_{d+1})})$ be an array (as defined in (2)) corresponding to the simplex V. Note that due to (9), condition (i) of Theorem 2.2 is satisfied, while (ii) holds trivially. Consequently, one can apply Theorem 2.2 to find a congruent copy of V in $\mathcal{K}$ and therefore property (iii) of Definition 3.1 is verified.

There is, however, as mentioned earlier, one more issue we need to clarify. By Definition 3.1, one needs to guarantee the existence of the family $\mathcal{H}(n)$ for all n sufficiently large. Unfortunately, the construction above applies only for some choices of n. Given $\vartheta = p/q$ recall that s and k were defined by Lemma 2.3 with $\eta = \vartheta/16$. Due to the choice of l which must be a multiple of $\omega = 8q(k+1)^{\binom{s}{k}-1}$, say $l = i\omega$, we infer that n is of the form $l(s+\vartheta(k+1)/8) = i\omega(s+\vartheta(k+1)/8) = iD$ for $D = \omega(s + \vartheta(k+1)/8)$. Observe also that the values of n for which the set $\mathcal{H}(n)$ satisfies Definition 3.1 form an infinite arithmetic progression $\{iD\}_{i=1}^{\infty}$. It remains to verify Definition 3.1 for all n sufficiently large. This will follow from the fact below.

FACT 3.10: Let c, α, and ε be fixed and let $\{iD\}_{i=1}^{\infty}$ be an infinite arithmetic progression. Let V be a finite set such that for every $i \geq 1$ there exists a set $\mathcal{H}(iD) \subseteq \mathbf{R}^{iD}$ satisfying (i)–(iii) of Definition 3.1. Then V is α-hyper Ramsey.

Proof of Fact 3.10: Fix some $\tilde{\varepsilon} < \varepsilon$ and choose i_0 sufficiently large such that

$$(1-\varepsilon)^{iD} \leq (1-\tilde{\varepsilon})^{(i+1)D} \tag{26}$$

for all $i \geq i_0$. Set $m_0 = i_0 D$. In order to prove that V is α-hyper Ramsey consider $m \geq m_0$ such that $iD < m < (i+1)D$ for $i \geq i_0$. We set $\mathcal{H}(m) = \mathcal{H}(iD) \subset S(\sqrt{\varrho(V)^2 + \alpha}, iD) \subset S(\sqrt{\varrho(V)^2+\alpha}, m)$. Since c is fixed, property (ii) of Definition 3.1 holds. Moreover, property (iii) of Definition 3.1 (with $\tilde{\varepsilon}$ instead of ε) follows from (26). ∎

We apply Fact 3.10 with $D = \omega(s + \vartheta(k+1)/8)$ and this finishes the proof of Lemma 3.9. ∎

3.4. ALMOST REGULAR SIMPLICES ARE α-HYPER RAMSEY SETS.

In this section we apply a result from [FR90] to show that almost regular simplices are α-hyper Ramsey. At first we define almost regular (i.e., (μ, β)-regular) simplices.

Definition 3.11: Let $1 \geq \mu \geq 0$ and $\beta > 0$ be given reals. A simplex $T = \{t_1, t_2, \dots, t_{d+1}\}$ is called **(μ, β)-regular** if, for every $1 \leq i < j \leq d+1$,

$$\beta(1-\mu) \leq \mathrm{d}^2(t_i, t_j) \leq \beta(1+\mu).$$

The following lemma was proved in [FR90] (cf. Lemma 3.1 in [FR90]).

LEMMA 3.12: *For every integer $d \geq 1$ there exists a real $1 \geq \mu = \mu(d+1) > 0$ such that, for every (μ, β)-regular simplex $T = \{t_1, t_2, \ldots, t_{d+1}\}$, there exists a $\binom{d+1}{2}$-dimensional box (i.e., the vertex set of a rectangular parallelepiped) P such that there exists a subset $T' \subseteq P$ congruent to T.*

Due to the fact that any two vertices of T' (from the lemma above) are not more than $\beta(1+\mu)$ apart, we can assume without loss of generality that each edge of the box is not longer than $\beta(1+\mu)$. Therefore, without loss of generality we only consider boxes P with circumradius

$$\varrho(P) \leq \frac{1}{2}\sqrt{\binom{d+1}{2}\beta^2(1+\mu)^2} = \frac{\beta(1+\mu)}{2}\sqrt{\binom{d+1}{2}}.$$

Since, due to Definition 3.11, $\mu \leq 1$, we infer that

$$\varrho(P) \leq \beta\sqrt{\binom{d+1}{2}} < \beta(d+1).$$

Combining this observation with Lemma 3.12 and Theorem 3.2 we derive the following:

LEMMA 3.13: *For every integer $d \geq 1$ there exists $\mu = \mu(d+1) > 0$ such that every (μ, β)-regular simplex $T = \{t_1, t_2, \ldots, t_{d+1}\}$ with circumradius $\varrho(T) = \varrho^T$ is α-hyper Ramsey for every $\alpha \geq \beta^2(d+1)^2 - (\varrho^T)^2$.*

4. Proof of the main result

In this section we prove the main result, Theorem 1.6, by proving the stronger statement, Theorem 3.3. We first outline the idea of the proof.

Given a simplex X and $\alpha > 0$, we construct a "smaller" simplex Z and a regular simplex $\widetilde{Z}$ such that $X \subseteq Z * \widetilde{Z}$. Then we find an α^V-hyper Ramsey simplex V which is "ϑ-close" to Z (see Lemma 3.9). Furthermore, we define a simplex T such that $V * T$ contains a subset X' congruent to X. Since V is very close to Z, T will be very close to $\widetilde{Z}$, and the right choice of constants will ensure that T is almost regular. Therefore, we will derive, by Lemma 3.13, that T is α^T-hyper Ramsey for some appropriate α^T. Finally, the product result, Lemma 3.4, will yield that X is $(\alpha^V + \alpha^T)$-hyper Ramsey with $\alpha^V + \alpha^T \leq \alpha$. Since $\alpha > 0$ was arbitrary, X is hyper Ramsey.

Proof of Theorem 3.3: Let $X = \{x_1, x_2, \ldots, x_{d+1}\}$ be a simplex and $1 > \alpha > 0$ be given. Without loss of generality assume that $\varrho(X) = 1$ and set

$$m_{ij} = \mathrm{d}^2(x_i, x_j).$$

As X is an affine independent set, we infer from Schoenberg's theorem, Theorem 2.1, that there exists a real $\gamma > 0$ such that the left-hand side of (1) is always less than $-\gamma$. Let

$$0 < \beta < \min\left\{\frac{\gamma}{(d+1)^2}, \sqrt{\frac{\alpha}{2(d+1)^2}}\right\} \tag{27}$$

be a sufficiently small real number (one additional upper bound on β will be stated later, after Remark 4.1). Then the matrix $M' = (m'_{ij})_{i,j=1}^{d+1}$ with $m'_{ij} = m_{ij} - \beta$ is of strictly negative type (by our choice of β in (27)) and thus, again by Theorem 2.1, there exists a simplex $Z = \{z_1, z_2, \ldots, z_{d+1}\} \subseteq \mathbf{R}^d$ such that for $1 \le i < j \le d+1$,

$$\mathrm{d}^2(z_i, z_j) = m'_{ij} = m_{ij} - \beta. \tag{28}$$

Remark 4.1: The regular simplex $\widetilde{Z}$ mentioned in the outline of the proof is the unique simplex with distance β between every two vertices. Due to the fact that we make no use of $\widetilde{Z}$, we don't explicitly mention it in the proof.

Moreover, assume we earlier choose β to be small enough such that

$$\varrho(Z) = \varrho^Z \le 1 + \alpha/8. \tag{29}$$

Let $\mu = \mu(d+1)$ be given by Lemma 3.13. Fix a small positive real ϑ by

$$\vartheta = \min\{\alpha, \beta\mu\} \tag{30}$$

and apply Lemma 3.9 for Z and ϑ. Consequently, we obtain an α^V-hyper Ramsey simplex $V = \{v_1, v_2, \ldots, v_{d+1}\}$ with

$$\alpha^V = (\varrho^Z)^2(1 + \vartheta/8) - (\varrho^V)^2$$

satisfying

$$\mathrm{d}^2(z_i, z_j) - \vartheta \le \mathrm{d}^2(v_i, v_j) \le \mathrm{d}^2(z_i, z_j) + \vartheta \tag{31}$$

for all $1 \le i < j \le d+1$, where ϱ^V equals the circumradius of V.

Finally, let $T = \{t_1, t_2, \ldots, t_{d+1}\}$ be the (last auxiliary) simplex defined by

$$\mathrm{d}^2(t_i, t_j) = m_{ij} - \mathrm{d}^2(v_i, v_j) \tag{32}$$

with circumradius $\varrho(T) = \varrho^T$. The simplex T is (μ, β)-regular by our choice of ϑ in (30). Indeed, by (28) and (31),

$$\beta - \vartheta \leq \mathrm{d}^2(t_i, t_j) \leq \beta + \vartheta$$

and hence

$$\beta(1 - \mu) \leq \mathrm{d}^2(t_i, t_j) \leq \beta(1 + \mu)$$

holds.

Thus, we may apply Lemma 3.13 and infer that T is α^T-hyper Ramsey for

$$\alpha^T = \beta^2(d+1)^2 - (\varrho^T)^2.$$

Now, Lemma 3.4 implies that $V * T$ is $(\alpha^V + \alpha^T)$-hyper Ramsey. Consequently, there exists an integer m_0^{V*T} and sets $\mathcal{H}^{V*T}(m)$ for $m \geq m_0^{V*T}$ such that

$$\begin{aligned}\mathcal{H}^{V*T}(m) &\subseteq S\Big(\sqrt{(\varrho^V)^2 + \alpha^V + (\varrho^T)^2 + \alpha^T}, m\Big)\\ &= S\Big(\sqrt{(\varrho^Z)^2(1+\vartheta/8) + \beta^2(d+1)^2}, m\Big).\end{aligned} \tag{33}$$

By (27), (29) and (30) we infer

$$(\varrho^Z)^2\Big(1 + \frac{\vartheta}{8}\Big) + \beta^2(d+1)^2 \leq \Big(1 + \frac{\alpha}{8}\Big)^3 + \frac{\alpha}{2} \leq 1 + \alpha,$$

which implies that

$$\mathcal{H}^{V*T}(m) \subseteq S(\sqrt{1+\alpha}, m+1).$$

On the other hand, it is easy to see that $V * T$ contains a subset X' congruent to X. In fact, setting $X' = \{x_i' = v_i * t_i \colon i = 1, 2, \ldots, d+1\}$ yields by (32) that

$$\mathrm{d}^2(x_i', x_j') = \mathrm{d}^2(v_i, v_j) + \mathrm{d}^2(t_i, t_j) = m_{ij} = \mathrm{d}^2(x_i, x_j),$$

which implies that $X' \subseteq V * T$ is congruent to X. Combining this with (33) we infer that X is α-hyper Ramsey. Since $\alpha > 0$ was chosen arbitrarily, X is hyper Ramsey. ∎

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